Homework Set 2

1. Let $C = V(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2$ be the *nodal cubic curve* over a field $k$ of characteristic not equal to 2.

(a) Construct an isomorphism between $C - \{(0,0)\}$ and $\mathbb{A}^1 - \{\pm 1\}$.

(b) Show that the isomorphism in (a) extends to a surjective morphism $\mathbb{A}^1 \to C$ sending the two points $t = \pm 1$ to the point $(x, y) = (0, 0)$ (later we'll see that this morphism identifies $\mathbb{A}^1$ with the *normalization* of $C$).

(c) What happens in characteristic 2?

2. A (commutative) ring is called *reduced* if $\sqrt{(0)} = (0)$, *i.e.*, the only nilpotent element in the ring is 0. We say that a scheme $X$ is *reduced* if the local ring $\mathcal{O}_{X,x}$ is reduced for every point $x \in X$. Show that an affine scheme $X = \operatorname{Spec}(R)$ is reduced if and only if $R$ is a reduced ring.

3. Let $X = \operatorname{Spec}(R)$ be a reduced affine scheme, and let $f \in R$, so the canonical ring homomorphism $R \to R[f^{-1}]$ corresponds to the inclusion of the basic open set $X_f \hookrightarrow X$. Show that the following are equivalent: (i) $R \to R[f^{-1}]$ is injective, (ii) $X_f$ is dense in $X$, (iii) no irreducible component of $X$ is contained in $V(f)$. What changes if $X$ is not assumed to be reduced?

4. Let $k$ be an algebraically closed field. Then all non-degenerate homogeneous quadratic forms $f(x_0, \ldots, x_n)$ over $k$ are similar up to a linear change of coordinates. The projective variety $Q_{n-1} = V(f) \subseteq \mathbb{P}^n_k$ is a called *quadric* of dimension $n - 1$.

(a) Verify that if we take $f = x_0x_1 + x_2^2 + \cdots + x_n^2$, then the intersection $Q_{n-1} \cap U_0$ of $Q_{n-1}$ with the standard open affine subset $U_0 = \mathbb{P}^n - V(x_0) \cong \mathbb{A}^n$ is the graph of a morphism $\mathbb{A}^{n-1} \to \mathbb{A}^1$, hence isomorphic to $\mathbb{A}^{n-1}$.

(b) Assume $n > 2$. Identifying $V(x_0) \subseteq \mathbb{P}^n$ with $\mathbb{P}^{n-1}$, show that the complement $Q_{n-1} \cap V(x_0)$ of $Q_{n-1} \cap U_0$ is a degenerate quadric, isomorphic to the cone over $Q_{n-3} \subseteq \mathbb{P}^{n-2} \subseteq \mathbb{P}^{n-1}$ with a point $p \in \mathbb{P}^{n-1}$ not on $\mathbb{P}^{n-2}$.

(c) Deduce that if $n$ is even, then $Q_{n-1}$ has a decomposition into $n$ disjoint locally closed subvarieties isomorphic to $\mathbb{A}^k$, one for each $k = 0, \ldots, n-1$. If $n$ is odd, then $Q_{n-1}$ has a decompostion into $n + 1$ affine spaces, now with two for $k = (n-1)/2$, and one for each of the the other integers $0 \leq k \leq n - 1$.

For those who know something about semi-simple algebraic groups or Lie groups: the quadric $Q_{n-1}$ is a quotient $G/P$, where $G$ is the algebraic group $SO_{n+1}(k)$ (which is also a Lie group if $k = \mathbb{C}$), and $P$ is a suitable maximal *parabolic subgroup*. The affine cells in the decomposition in (c) are the *Schubert cells*. The different pictures in the even and odd cases reflect the fact that in the classification of semi-simple groups, the groups $SO_{n+1}(k)$ belong to two different families depending on whether $n$ is even or odd.

5. In class we proved the representability theorem, that if a functor $F : \textbf{Schemes}^{\mathrm{op}} \to \textbf{Sets}$ is a sheaf in the Zariski topology, and has a covering by representable open subfunctors, then $F$ is representable, *i.e.*, $F \cong \underline{X}$ for a scheme $X$.

Adapt the definitions, the statement of the theorem, and the proof to the situation where we replace the category of all schemes with the category $\textbf{Schemes}/S$ of schemes over a fixed base scheme $S$.

6. Prove that a non-empty scheme $X$ has the property that every open covering $X = \bigcup_\alpha U_\alpha$ is trivial (*i.e.*, one of the $U_\alpha$ is equal to $X$) if and only if $X = \operatorname{Spec} R$, where $R$ is a local ring.

7. By the theorem on the functor represented by an affine scheme, if $T = \operatorname{Spec}(R)$ and $X$ is any locally ringed space, then every ring homomorphism $\alpha\colon R \to \mathcal{O}(X)$ arises from a unique morphism of locally ringed spaces $\phi\colon X \to T$. Show that the uniqueness fails if we allow $\phi$ to be an arbitrary morphism of ringed spaces. Hint: any example of a non-local morphism of ringed spaces $X \to T$, where $X$ is a locally ringed space and $T$ is an affine scheme, will do the job.

8. Prove that for every scheme $X$ there is an affine scheme $Y$ and a morphism $\pi\colon X \to Y$ such that every morphism from $X$ to an affine schem factors uniquely through $Y$ (actually, this works for any locally ringed space $X$).

9. Let $Z$ be an object in a category $\mathbf{C}$, and let $\underline{Z}(-) = \operatorname{Hom}(-, Z)$ be the functor $\mathbf{C}^{\mathrm{op}} \to \textbf{Sets}$ represented by $Z$. According to Yoneda's Lemma, for any functor $F\colon \mathbf{C}^{\mathrm{op}} \to \textbf{Sets}$, there is a canonical bijection between functorial maps from $\underline{Z}$ to $F$ and elements $f \in F(Z)$. More explicitly, if $\alpha\colon \underline{Z} \to F$ is a functorial map, then $f = \alpha_Z(1_Z)$, where $1_Z \in \underline{Z}(Z)$ is the identity arrow on $Z$.

(a) Supposing $f \in F(Z)$ is given, describe the corresponding functorial map $\alpha$, and verify that $\alpha_T\colon \underline{Z}(T) \to F(T)$ is in fact functorial in $T$.

(b) Verify that the Yoneda correspondence is functorial in $Z$.

(c) Verify that the Yoneda correspondence is functorial in $F$.

Part of the exercise is to figure out precisely what the relevant functorialities mean.

10. A topological space $X$ is *Noetherian* if every strictly decreasing chain of closed subsets in $X$ is finite.

(a) Prove that $X$ is Noetherian if and only if every open subset of $X$ is quasi-compact.

(b) Prove that if $R$ is a Noetherian ring then $\operatorname{Spec}(R)$ is a Noetherian space.

(c) Prove that every Noetherian space has finitely many irreducible components.

(d) Deduce that every Noetherian ring has finitely many minimal prime ideals.

(e) Show that the converse to (b) does not hold.

11. Let $A = k[x, y]$, where $k$ is an algebraically closed field.

(a) Show that if two polynomials $p(x, y)$, $q(x, y)$ have no common factor, then the solution set $V(p, q) \subseteq k^2$ is finite.

(b) Deduce that every prime ideal of $A$ is one of the following: (i) the zero ideal, (ii) a maximal ideal $(x-a, y-b)$ for some $(a,b) \in k^2$, or (iii) a principal ideal $(f)$ generated by an irreducible polynomial $f(x,y)$. You will need Hilbert's nullstellensatz. Note that this result amounts to a description of all the irreducible closed subvarieties of $k^2$.

12. With $A$ as in the previous problem, let $f(x,y) \in A$ be an irreducible polynomial and $B = k[x,y]/(f)$.

(a) Show that the points of $\operatorname{Spec}(B)$ are (i) the generic point $\mathfrak{p} = (0)$, and (ii) closed points corresponding to points $(a,b)$ on the curve $V(f)$ in $k^2$.

(b) Show that the proper closed sets of $X = \operatorname{Spec}(B)$ are just the finite sets of closed points. In particular, all irreducible curves $V(f)$ in $k^2$ are homeomorphic in the Zariski topology, although they need not be isomorphic as schemes.

13. Let $k$ be a commutative ring. Let $R$ be a polynomial ring over $k$ in $n^2$ variables $x_{ij}$, $1 \leq i,j \leq n$. Thinking of the $x_{ij}$ as the entries of an $n \times n$ matrix $M$, let $d = \det(M)$ and let $A = R[d^{-1}]$, so $\operatorname{Spec}(A)$ is the affine open subset $\mathbb{A}_k^{(n^2)} - V(d)$. Define $GL_n = \operatorname{Spec}(A)$.

(a) Prove that for any scheme $T$ over $k$, the set $GL_n(T)$ of $k$-morphisms $T \to GL_n$ is canonically identified with the set of invertible $n \times n$ matrices over $\mathcal{O}_T(T)$.

(b) Prove that there are unique morphisms $m\colon GL_n \times_k GL_n \to GL_n$, $e\colon \operatorname{Spec}(k) \to GL_n$ and $i\colon GL_n \to GL_n$ so that for every $k$-scheme $T$, the maps $GL_n(T) \times GL_n(T) \to GL_n(T)$, $\{\text{point}\} \to GL_n(T)$, and $GL_n(T) \to GL_n(T)$ induced by $m$, $e$ and $i$ give the group law, unit element, and inverse in the group of invertible $n \times n$ matrices over $\mathcal{O}_T(T)$.

(c) Show that the morphism $m$ does *not* in general define a group law on the underlying set of the scheme $GL_n$, not even in the simplest case, where $k$ is a field and $n = 1$.

14. Let $X$ be a disconnected scheme, that is, $X$ is the disjoint union of two non-empty open (and therefore closed) subschemes $X_1$ and $X_2$. Prove that the ring $\mathcal{O}_X(X)$ is the Cartesian product $\mathcal{O}_{X_1}(X_1) \times \mathcal{O}_{X_2}(X_2)$. Conversely, prove that that if $X$ is a scheme and $\mathcal{O}_X(X)$ is a Cartesian product $A_1 \times A_2$, with neither ring $A_i$ the zero ring, then $X$ is disconnected.

15. The set $X$ of $m \times n$ matrices of rank $\leq r$ over an algebraically closed field $k$ is a classical affine variety in $\mathbb{A}_k^{(mn)}$, defined by the vanishing of all $(r+1) \times (r+1)$ minors of the matrix of coordinates $x_{ij}$ on $\mathbb{A}_k^{(mn)}$.

(a) Find a surjective morphism from an affine space onto $X$. This shows that $X$ is irreducible.

(b*) Prove that the $(r+1) \times (r+1)$ minors of the matrix of coordinates generate the ideal $\mathcal{I}(X)$. Hint: let $I \subseteq R = k[x_{1,1}, \ldots, x_{m,n}]$ be the ideal generated by these minors. The problem is to prove that $I$ is a prime ideal. Use the morphism in (a) to construct a ring homomorphism $\phi\colon R/I \to S$, where $S$ is a polynomial ring. To prove that $\phi$ is injective, find a set of monomials $M$ in $R$ such that $M$ spans $R/I$ as a vector space, and $\phi(M)$ is linearly independent in $S$.